

GLOBAL MINIMALITY OF GENERIC MANIFOLDS AND HOLOMORPHIC EXTENDIBILITY OF CR FUNCTIONS

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Introduction.

Let M be a smooth generic submanifold of \mathbf{C}^n . Several authors have studied the property of CR functions on M to extend locally to manifolds with boundary attached to M and holomorphically to generic wedges with edge M (cf. [14], [67], [68]). In a recent work ([69]), Tumanov has showed that CR-extendibility of CR functions on M propagates along curves that run in complex tangential directions to M . His main result appears as a natural generalization of results by Trépreau on propagation of singularities of CR functions ([61]). Indeed, Theorem 5.1 in [69] states that the direction of CR-extendibility moves parallelly with respect to a certain differential geometric partial connection in a quotient bundle of the normal bundle to M , and this variation is dual to the one introduced by Trépreau, according to Proposition 7.3 in [69].

In this paper we give a new and simplified presentation of the connection introduced in Tumanov's work. Let M be a real manifold and N a submanifold of M , K a subbundle of TM with the property that $K|_N \subset TN$. Then by means of the Lie bracket, we can define a K -partial connection on the normal bundle of N in M (Proposition 1.1). In general, the parallel translation associated with that partial connection will be induced by the flow of K -tangent sections of TM (Proposition 1.2). When M is a generic submanifold of \mathbf{C}^n containing a CR submanifold S with the same CR dimension we recover in section 2 the $T^c S$ -partial connection constructed by Tumanov in [69].

Recall that the *CR-orbit* of a point $z \in M$ is the set of points that can be reached by piecewise smooth integral curves of complex tangent vector fields. We then say that M is *globally minimal* at a point $z \in M$ if the CR-orbit of z contains a neighborhood of z in M . Using previous results, we show that vector space generated by the directions of CR-extendibility of CR functions on M exchanges by the induced composed flow between two points in a same CR-orbit (Lemma 3.5). As an application, we prove the main result of this paper, conjectured by Trépreau in [61] : *for wedge extendibility of CR functions to hold at every point in the CR-orbit of $z \in M$ it is sufficient that M be globally minimal at z* (Theorem 3.4). Up till now we can only conjecture the converse (for a local result, see [6]).

I wish to thank J.-M. Trépreau for helpful critical and simplifying remarks.

Remark : After this paper was completed, we have received a preprint by B. Jöricke *Deformation of CR-manifolds, minimal points and CR-manifolds with the microlocal analytic extension property*, which contains also a proof of Theorem 3.4 and Theorem 3.6. Our proof seems quite different since we obtain these results relying on Tumanov's propagation theorems, the generic manifold M being fixed, whereas B. Jöricke works with conic perturbations of the base manifold so as to produce minimal points.

§1. Partial connections associated with a system of vector fields.

Let M be a real differentiable manifold of class C^2 of dimension n and $H \rightarrow M$ a r -dimensional vector bundle over M . Recall that a connection ∇ on the bundle $H \rightarrow M$ is a bilinear mapping which assigns to each pair of a vector field X with domain U and a section η of H over U a section $\nabla_X \eta$ of H over U and satisfy

$$\nabla_{\phi X} = \phi \nabla_X, \quad \nabla_X(\phi \eta) = \phi \nabla_X \eta + (X\phi)\eta, \quad \phi \in C^1(M, \mathbf{R}).$$

When the covariant derivative $\nabla_X \eta$ can only be defined for vectors X that belong to a subbundle K of TM , we call the connection ∇ a K -partial connection (cf. [69]).

If N is a submanifold of M , let $T_N M$ be the *normal bundle of N in M* , i.e.

$$T_N M = TM|_N / TN.$$

PROPOSITION 1.1. *Let M be a real manifold of class C^2 , $N \subset M$ a submanifold of class C^2 too and let K be a C^1 subbundle of TM with the property that $K|_N \subset TN$. Then there exists a natural K -partial connection ∇ on the bundle $T_N M$ which is defined as follows. If $x \in N, X \in K[x]$ and η is a local section of $T_N M$ over a neighborhood of x , then take*

$$\nabla_X \eta = [\tilde{X}, \tilde{Y}](x) \mod T_x N$$

where \tilde{X} is a C^1 local section of K extending X and \tilde{Y} is a lifting of η in TM in a neighborhood of x .

PROOF. We first check that the definition is independent of the lifting \tilde{Y} . In fact, when \tilde{Y} is tangent to N , as \tilde{X} is tangent to N too, the Lie bracket $[\tilde{X}, \tilde{Y}]$ remains tangent to N hence is zero in the quotient bundle.

Next we have to check that the definition of ∇ is independent of the chosen section \tilde{X} or, to rephrase, that if $\tilde{X}(x) = 0$ then $[\tilde{X}, \tilde{Y}](x)$ belongs to $T_x N$. Since K is a fiber bundle we can write

$$\tilde{X} = \sum_{j=1}^r f_j \tilde{X}_j \quad f_j(0) = 0 \quad j = 1, \dots, r$$

where $r = \text{rank } K$, $(\tilde{X}_j)_{j=1, \dots, r}$ is a frame for K near x and the f_j are C^1 real valued functions defined near x . Noting that

$$[f\tilde{X}, \tilde{Y}] \equiv f[\tilde{X}, \tilde{Y}] - (\tilde{Y}f)\tilde{X} \equiv f[\tilde{X}, \tilde{Y}] \mod TN$$

the result follows and the mapping ∇ is well-defined. Moreover the preceding implies that if $\phi \in C^1(M, \mathbf{R})$

$$\nabla_{\phi X} \eta \equiv \phi \nabla_X \eta.$$

Last, we check that ∇_X is a derivation. Indeed

$$\nabla_X(\phi \eta) \equiv [\tilde{X}, \phi \tilde{Y}](x) \equiv (\tilde{X} \cdot \phi) \tilde{Y} + \phi [\tilde{X}, \tilde{Y}] \equiv (X\phi)\eta + \phi \nabla_X \eta$$

and the proof is complete.

With the connection ∇ it is associated the *parallel translation* of fibers of $T_N M$ along smooth curves on the base N that run in directions tangent to K . Let $I \ni t$ be a subinterval of \mathbf{R} and $\gamma : I \rightarrow N$ be a smooth curve with the property that $\dot{\gamma}(t) \in K[\gamma(t)]$, where $\dot{\gamma} = \frac{d}{dt}\gamma(t)$. A curve $\eta(t) \in T_N M[\gamma(t)]$ is a *horizontal lift* of γ if $\nabla_{\dot{\gamma}}\eta = 0$. Existence and uniqueness of horizontal lifts provide linear isomorphisms

$$\Phi_{t_0, t} : T_N M[\gamma(t_0)] \rightarrow T_N M[\gamma(t)]$$

obtained by moving elements of K along horizontal lifts of γ .

Recall (cf. [58]) that the Lie bracket $[\tilde{X}, \tilde{Y}]$ is defined as the Lie derivative $L_{\tilde{X}}\tilde{Y}$ of \tilde{Y} with respect to \tilde{X}

$$[\tilde{X}, \tilde{Y}](x) = L_{\tilde{X}}\tilde{Y} = \lim_{h \rightarrow 0} [\tilde{Y}(x) - d\tilde{X}_{-h}(\tilde{Y}(\tilde{X}_h(x)))]$$

where \tilde{X}_t is the local flow on M generated in a neighborhood of x by the vector field \tilde{X} , and $d\tilde{X}_t$ denotes its differential. In the assumptions of Proposition 1.1, \tilde{X} is of class C^1 so the mapping $x \rightarrow \tilde{X}_t(x)$ is of class C^1 and the differential is a well-defined continuous mapping. When \tilde{X} is K -tangent its flow (and more generally any piecewise smooth composition of such flows) stabilizes the tangent bundle TN of the manifold N , hence its differential induces isomorphisms of fibers of $T_N M$, which we denote by dX_t . Assume moreover that the curve γ is an integral curve of a C^1 K -tangent vector field \tilde{X} , (which cannot be true for most general smooth curves γ but is sufficient enough for the applications) : $\gamma(0) = x$ and $\gamma(t) = \tilde{X}_t(x)$. Then we claim that the mapping

$$dX_t : T_N M[x] \rightarrow T_N M[\gamma(t)]$$

provides the parallel translation $\Phi_{0, t}$. Indeed let $\eta_0 \in T_N M[x]$ and take $\eta(t) = dX_t(\eta_0)$. Then by the definition of the partial connection ∇ and the definition of the Lie bracket we have

$$\nabla_{\dot{\gamma}}\eta(t) \equiv 0.$$

By uniqueness of solutions of linear differential equations of order one it must be that

$$\eta(t) = \Phi_{0, t}(\eta_0).$$

PROPOSITION 1.2. *Under the hypotheses of Proposition 1.1, let $\gamma(t) = X_t(x_1)$ be a smooth (piecewise smooth) integral curve of a K -tangent vector field X (a finite number of K -tangent vector fields) running from $x_1 \in N$ to $x_2 \in N$. Then the parallel translation along γ associated with the K -partial connection ∇ is induced by the differential of the flow of X (composed flow).*

In order to give an expression of the covariant derivatives induced by the partial connection ∇ , we choose coordinates on M , $x = (x', x'') \in \mathbf{R}^l \times \mathbf{R}^m$ such that the base point corresponds to $x = 0$ and the submanifold N is defined by the equation $x'' = 0$. Let $(x, \eta) = (x', x'', \eta', \eta'')$ be the canonical coordinates on TM , and $(x', \eta'') \in \mathbf{R}^l \times \mathbf{R}^m$ the associated coordinates on $T_N M$.

If $X = \sum_{j=1}^{l+m} a_j(x) \frac{\partial}{\partial x_j}$ is a C^1 section of K , it must be tangent to N , so $a_j(x', 0) = 0$, $j = l+1, \dots, l+m$. We choose a local section η of $T_N M$ over a neighborhood of 0 in N , in fact a section \tilde{Y} of TM of the form

$$\tilde{Y} = \sum_{j=l+1}^{l+m} \eta_j(x') \frac{\partial}{\partial x_j},$$

Recalling Proposition 1.1 we have the following expression for the covariant derivative of η in the direction of X

$$\nabla_X \eta = \sum_{j=l+1}^{l+m} \sum_{k=1}^l a_k(x', 0) \frac{\partial \eta_j}{\partial x_k}(x') \frac{\partial}{\partial x_j} - \sum_{j,k=l+1}^{l+m} \eta_k(x') \frac{\partial a_j}{\partial x_k}(x', 0) \frac{\partial}{\partial x_j}.$$

Given an integral curve $\gamma(t) = (\gamma'(t), 0)$ of the field X , the equations for the horizontal lifts look like

$$X.\eta_j = \dot{\eta}_j(t) = \sum_{k=l+1}^{k=l+m} \frac{\partial a_j}{\partial x_k}(\gamma'(t), 0) \eta_k(t) \quad j = l+1, \dots, l+m$$

so the curve $(\gamma'(t), \eta''(t))$ is the integral curve of the following vector field \tilde{X} on $T_N M$

$$\tilde{X}(x', \eta'') = \sum_{j=1}^l a_j(x', 0) \frac{\partial}{\partial x_j} + \sum_{j,k=l+1}^{l+m} \frac{\partial a_j}{\partial x_k}(x', 0) \eta_k(x') \frac{\partial}{\partial \eta_j}.$$

Alternately, the partial connection ∇ can be defined by the family of horizontal subspaces $H(\eta) \subset T_\eta(T_N M)$ generated by vectors of the form \tilde{X} .

Let us consider the *dual connection* ∇^* to the connection ∇ on the dual bundle $T_N^* M$. Recall that the conormal bundle of N in M , $T_N^* M$, consists of forms in $T^* M$ that vanish on TN . It has fiber over a point $x \in N$

$$T_N^* M[x] = \{\phi \in T_x^* M; \phi|_{T_x N} = 0\}.$$

The dual connection ∇^* is defined by the following relation : if X is a K -tangent vector to N at x , η is any section of $T_N M$ near x and ϕ is any section of $T_N^* M$

$$X \langle \phi, \eta \rangle = \langle \nabla_X^* \phi, \eta \rangle + \langle \phi, \nabla_X \eta \rangle.$$

It is easily checked that such a relation defines a K -partial connection on $T_N^* M$.

Along with the coordinates on $T_N M$ we introduced before we can introduce the canonical coordinates (x', ξ'') on the conormal bundle $T_N^* M$. These are dual to the coordinates (x', η'') for the canonical duality \langle, \rangle between $T_N M$ and $T_N^* M$:

$$\langle \sum_{j=l+1}^{l+m} \xi_j dx_j, \sum_{j=l+1}^{l+m} \eta_j \frac{\partial}{\partial x_j} \rangle = \sum_{j=l+1}^{l+m} \xi_j \eta_j.$$

Using the previous definition of the dual connection we can then compute the covariant derivative of a section $\sum \xi_j dx_j = \phi$ of $T_N^* M$. One easily shows

$$\nabla_X^* \phi = \sum_{j=l+1}^{l+m} (X.\xi_j + \sum_{k=l+1}^{l+m} \xi_k \frac{\partial a_k}{\partial x_j}) dx_j.$$

Hence, under the assumption of Proposition 1.2, the parallel translation associated with the connection ∇^* is given by means of the integral curves of the following vector field on $T_N^* M$

$$\hat{X} = \sum_{j=1}^l a_j(x', 0) \frac{\partial}{\partial x_j} - \sum_{j,k=l+1}^{l+m} \frac{\partial a_j}{\partial x_k}(x', 0) \xi_j \frac{\partial}{\partial \xi_k}.$$

There is another way of thinking the connection ∇^* dual to the partial connection ∇ which has been considered by Trépreau in [61].

To a general vector field X on M it is associated its symbol $\sigma(X)$ which is an invariantly defined function on the cotangent bundle T^*M of M . To a function f of class C^1 on T^*M it is associated its hamiltonian field H_f .

Let X_j , $j = 1, \dots, r$ be a local basis of K -tangent sections of TM . Let Σ_K be the orthogonal complement of K in T^*M . If $X = \sum_{j=1}^r \phi_j X_j$ is a C^1 section of K we have

$$H_{\sigma(X)}|_{\Sigma_K} = \sum_{j=1}^r \phi_j H_{\sigma(X_j)}|_{\Sigma_K} + \sum_{j=1}^r \sigma(X_j) H_{\phi_j}|_{\Sigma_K}.$$

Since $\sigma(X_j)$, $j = 1, \dots, r$ is zero on Σ_K , we deduce that the restricted hamiltonian field

$$H_{\sigma(X)}|_{\Sigma_K} = \sum_{j=1}^r \phi_j H_{\sigma(X_j)}|_{\Sigma_K}$$

depends only on the value of X at the base point and not on the chosen section. If X is tangent to N , $H_{\sigma(X)}$ when restricted to T_N^*M is tangent to T_N^*M . Hence we have constructed another vector field on T_N^*M which is in fact the same as the one associated with the connection dual to the partial connection ∇ .

Indeed, let as before (x', ξ'') be the canonical coordinates on the conormal bundle T_N^*M . Recall that the hamiltonian field of a function $f = f(x, \xi)$ just looks like

$$H_f = \sum_{j,k=1}^{j,k=l+1} \frac{\partial f}{\partial \xi_k} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_k}.$$

The symbol of the section

$$X = \sum_{j=1}^{l+m} a_j \frac{\partial}{\partial x_j} \quad a_j(x', 0) = 0, \quad j = l+1, \dots, l+m$$

of K being $\sigma(X) = \sum a_j \xi_j$ we can compute

$$H_{\sigma(X)}|_{T_N^*M} = \sum_{j=1}^l a_j(x', 0) \frac{\partial}{\partial x_j} - \sum_{j,k=l+1}^{l+m} \frac{\partial a_j}{\partial x_k}(x', 0) \xi_j \frac{\partial}{\partial \xi_k}$$

and the last expression proves that $H_{\sigma(X)}$ is the same vector field on T_N^*M as \hat{X} computed previously, so the set of restricted hamiltonian fields $H_{\sigma(X)}|_{T_N^*M}$ defines the same family of horizontal subspaces for the partial connection ∇^* .

The next section is devoted to the application of the preceding results to the geometry of CR submanifolds of \mathbf{C}^n .

§2. Application to generic submanifolds of \mathbf{C}^n

In this section we apply results of section 1 in the context of differential geometry in the complex euclidean space \mathbf{C}^n . Afterwards we check that our definitions recover those of Trépreau [61] and Tumanov [62].

Let $T\mathbf{C}^n$ be the real tangent bundle of \mathbf{C}^n and J be the standard complex structure operator on $T\mathbf{C}^n$. Let $T^*\mathbf{C}^n$ be the bundle of *holomorphic* (\mathbf{C} -linear) 1-forms on \mathbf{C}^n . In the canonical coordinates $z = (z_1, \dots, z_n)$ its fiber over a point z consists of $(1,0)$ -forms $\omega = \sum_{j=1}^n \zeta_j dz_j$, $\zeta_j \in \mathbf{C}$, $j = 1, \dots, n$. Then

$T^*\mathbf{C}^n$ is a *complex* manifold. It can be (and it is usually) identified with the real dual bundle of $T\mathbf{C}^n$ introducing the real duality defined by

$$(\omega, X) \in T^*\mathbf{C}^n \times T\mathbf{C}^n \quad (\omega, X) \longmapsto \text{Im} \langle \omega, X \rangle.$$

In other words we identify real and holomorphic forms by $\text{Im} \omega \leftrightarrow \omega$.

Now, let M be a real submanifold of \mathbf{C}^n . In this identification, the conormal bundle $T_M^*\mathbf{C}^n$ is a subbundle of $T^*\mathbf{C}^n$ and it has fiber spaces

$$T_M^*\mathbf{C}^n[z] = \{ \omega \in T^*\mathbf{C}^n; \text{Im} \omega|_{T_z M} = 0 \}.$$

Hence the bundles $T_M\mathbf{C}^n = T\mathbf{C}^n|_M/TM$ and $T_M^*\mathbf{C}^n$ are in duality by

$$(\omega, X \bmod TM) \longmapsto \text{Im} \langle \omega, X \rangle.$$

Assume moreover that M is generic (that is $TM + JTM = T\mathbf{C}^n|_M$) and let Σ_M be the orthogonal complement of the complex tangent bundle T^cM in the cotangent bundle T^*M . In the terminology of linear partial differential equations it is the *characteristic set* (and since T^cM is a fiber bundle, the characteristic *manifold*) of the system of CR vector fields. It is easily checked that Σ_M and TM/T^cM are in duality in the same way.

Since M is CR, Σ_M is a fiber bundle and there is a canonical bundle epimorphism

$$\theta : T_M^*\mathbf{C}^n \rightarrow \Sigma_M,$$

defined by $\theta(\omega) = \iota_M^* \omega$ where $\iota_M : M \rightarrow \mathbf{C}^n$ is the natural injection. Since M is generic, θ is an isomorphism.

On the other hand the complex structure J induces an isomorphism, still denoted by J

$$J : TM/T^cM \rightarrow T_M\mathbf{C}^n.$$

LEMMA 2.1. θ is the transposed of J , i.e.

$$(\omega, JX) = (\theta(\omega), X)$$

for every ω, X .

(Indeed $\langle \omega, JX \rangle = i \langle \omega, X \rangle$).

From now on we let $S \subset M$ be a CR submanifold of M with the property that $CRdim S = CRdim M$. Equivalently it is required that $T^cS = T^cM|_S$. By restriction analogous pairs of bundles remain isomorphic when TM/T^cM is replaced by $T_S M$, Σ_M is replaced by $T_S^* M$, $T_M\mathbf{C}^n$ is replaced by $T\mathbf{C}^n|_S / (TM|_S + JTS) = E$, and $T_M^*\mathbf{C}^n$ is replaced by $T_M^*\mathbf{C}^n \cap iT_N^*\mathbf{C}^n = E^*$, but now T^cS -partial connections can be defined by means of the isomorphisms J and θ on the two new bundles E and E^* . Note that the duplication essentially deals with complex differential geometry.

First, the results of the previous section apply with $K = T^cM = TM \cap JTM$ and $N = S$ and produce a T^cS -partial connection ∇ on $T_S M$ together with the dual connection ∇^* on $T_S^* M$. On the other hand, the push forward by J of ∇ defines a T^cS -partial connection Θ on E ; its action on a section ϑ of E in the direction of a complex tangent vector X is simply

$$\Theta_X \vartheta := J \nabla_X (J^{-1} \vartheta).$$

Similarly, the pull-back of the T^cS -partial connection ∇^* by θ defines a T^cS -partial connection Θ^* on E^* , and Θ^* is the connection dual to Θ since θ is the transposed of J (lemma 2.1).

Recall from section 1 that if X is a section of T^cM then $\hat{X} = H_{\sigma(X)}|_{T_S^*M}$ is tangent to T_S^*M . In [61], Trépreau showed that E^* is a CR manifold, using a lemma which states that given such a vector field \hat{X} tangent to T_S^*M with horizontal part X complex tangent to M , there exists a unique vector field \tilde{X} complex tangent to E^* with the same horizontal part X . Moreover Trépreau states that

$$\hat{X} = d\theta(\tilde{X})$$

Hence we deduce that the T^cS -partial connection $\Theta^* = \theta^*\nabla^*$ can alternately be given, as is originally done in [61], by means of the vector fields of the form \tilde{X} , i.e. horizontal subspaces of Θ^* are spanned by tangent vectors to integral curves of \tilde{X} . We then have checked that the parallel translation in E^* introduced by Trépreau with the assumption of Proposition 1.2 is the same as the one associated with the T^cS -partial connection Θ^* previously defined starting, as in section 1, with the partial connection associated with the bundle of complex tangents to M , $K = T^cM$. Moreover, since \tilde{X} is complex tangent to E^* , we see that T^cE^* is the set of horizontal subspaces for the T^cS -partial connection Θ^* . This has been noticed in [69] and will be usefull in the next section when proving Theorem 3.4.

§3. Orbits and the extension of CR functions.

In this section, it is assumed that M is a generic submanifold of \mathbf{C}^n of smoothness class C^2 , and we let \mathbf{X} be the set of C^1 sections over open subsets of M of T^cM . If $z \in M$, the subset of M consisting of points of M which can be reached by piecewise C^1 -smooth integral curves of elements of \mathbf{X} , starting at z , is called the *CR-orbit* of z , and is denoted by $\mathcal{O}[z]$.

If U is an open subset of M , $\mathbf{X}|_U$ denotes the set of elements of \mathbf{X} restricted to U . It is well-known (cf. [59], [6], [61]) that

$$\lim_{\substack{\longrightarrow \\ U}} \mathcal{O}(\mathbf{X}|_U, \dagger)$$

where U runs over the open neighborhoods of z in M defines the germ at z of the unique CR-submanifold of M with the same CR dimension as M of minimal dimension passing through z , which is called the *local CR-orbit* of z and is denoted by $\mathcal{O}^{loc}[z]$. When considering $\mathcal{O}^{loc}[z]$ in the following we shall mean such a submanifold of a neighborhood of z in M , i.e. an actual representative of the germ. It plays the crucial role in the study of automatic extendibility of CR functions (cf. Theorem 3.1 below).

Recall that a smooth complex-valued function on M is called a *CR function* if it is annihilated by every antiholomorphic tangent vector field on M . A continuous function can be thought CR in the sense of distribution theory. We denote by $CR(M)$ the set of all continous CR functions on M .

For completeness we recall definitions from [61] and [69]. We say that a manifold \tilde{M} with boundary is *attached to M at (m, u)* , $m \in M$, $u \neq 0$, $u \in T_M\mathbf{C}^n[m]$ if $b\tilde{M} \cap U = M \cap U$ for some neighborhood U of m , and u is represented by a vector $u_1 \in T_m\tilde{M}$ directed inside \tilde{M} .

Let f be a CR function on M ; we say that f is *CR-extendible* at (m, u) if it extends continuously to be CR on some \tilde{M} attached to M at (m, u) . When there is a CR submanifold S of M through m and a manifold \tilde{M} attached to M at (m, u) , $u \in T_M\mathbf{C}^n[m]$, we also say that \tilde{M} is attached to M at (m, η) , if u represents $\eta \in E_m$, $\eta \neq 0$ (E is the bundle defined in section 2). Similarly it makes sense to consider CR-extendibility at (m, η) , $m \in S$, $\eta \in E_m$. But it should be noted that given $\eta \neq 0$ in E_m does not determine \tilde{M} unambiguously unless S is complex

From now on we will require that M belong to the class $C^{(k,\alpha)}$, $k \geq 2, 0 < \alpha < 1$. This regularity assumption can be justified since it behaves well when proving the strongest local results on CR-extendibility (In fact, it behaves well through the so-called *Bishop equation*, [68], Theorem 1.), and constructing wedges with ribs and an edge having such a regularity (cf. [4]). Moreover, we need manifolds of class at least C^2 in order to apply Proposition 1.1. Since it will be of use in the proof of Theorem 3.4 we recall the following theorem due to Tumanov ([68])

THEOREM 3.1. (A. E. TUMANOV) *Let M be a generic submanifold of $\mathbf{C}^n, n = p + q$, with $\dim M = 2p + q$, $\text{CRdim } M = p$, and of smoothness class $C^{k,\alpha}$ ($k \geq 2$), $0 < \alpha < 1$. For every point $z \in M$ there exist $r = r(z) = \dim \mathcal{O}_{[z]}^{\text{loc}} - 2$ $\text{CRdim } M$ manifolds with boundary $\tilde{M}_1, \dots, \tilde{M}_r$ attached to M at z , of class $C^{(k,\beta)}$ whenever $0 < \beta < \alpha$ such that*

(a) *Every CR function on M is CR-extendible to $\tilde{M}_1, \dots, \tilde{M}_r$*

(b) $\sum_{j=1}^r T_{z'} \tilde{M}_j = T_{z'} M + J T_{z'} \mathcal{O}_{[z]}^{\text{loc}} \quad z' \text{ close to } z \text{ in } \mathcal{O}_{[z]}^{\text{loc}}.$

Moreover the manifold germ $\mathcal{O}_{[z]}^{\text{loc}}$ is of class $C^{(k,\beta)}$ whenever $0 < \beta < \alpha$.

Note that $\mathcal{O}_{[z]}^{\text{loc}}$ is at least of class C^2 so it can play the role of N in Propositions 1 and 2. Using the connections constructed in section 2 we can reinterpret the main result on propagation of analyticity for CR functions recently proved by Tumanov.

According to Tumanov ([69], Proposition 7.3), the connection dual to the one that is constructed during the paper has the property that its horizontal subspaces are exactly fibers of the complex tangent bundle $T^c E^*$, hence, concludes Tumanov, the induced parallel translation need be the same as the one introduced on E^* by Trépreau. We have shown in section 2 that our connection Θ has as a dual connection a connection Θ^* with the same property; so $\Theta = J_* \nabla$ coincides with the connection constructed by Tumanov.

Proposition 1.2 together with Theorem 5.1 in [69] leads to

THEOREM 3.2. *Let $M \subset \mathbf{C}^n$ be a generic manifold and $S \subset M$ a CR submanifold of M with the property that $\text{CRdim } S = \text{CRdim } M$. Let γ be a piecewise smooth integral curve of $T^c M$ running from $z' \in S$ to $z'' \in S$ and let Φ_γ be the associated composed flow. Then for every $\epsilon > 0$, every $\eta' \in E_{z'}$ and every manifold \tilde{M}' attached to M at (z', η') , there exists another manifold \tilde{M}'' attached to M at (z'', η'') , $\eta'' \in E_{z''}$ such that*

(a) $|\eta'' - J d\Phi_\gamma(z).J^{-1}\eta'| < \epsilon$

(b) *if a CR function on M extends to be CR on \tilde{M}' it extends to be CR on \tilde{M}''*

(c) *if M, \tilde{M}' belong to $C^{k,\alpha}$ ($k \geq 2$), $0 < \gamma < \alpha < 1$ then there exists such a $\tilde{M}'' \in C^{(k,\gamma)}$.*

Theorem 3.2 shows that the so-called propagation of analyticity for CR functions is intrinsically related to the geometry of the base manifold M . Moreover, it fundamentally means that the study of extendibility for CR functions is closely related to the study of sections of the complex tangent space to M .

Following Sussmann ([59]), we begin with some adapted terminology and recalls. Let $X \in \mathbf{X}$ be a local section of $T^c M$. The C^1 integral curves $t \rightarrow \gamma(t)$ of X generate local diffeomorphisms of M where they are defined (the so-called *flow* of X) which we will denote by $z \rightarrow X_t z$. Composites of several maps of the form X_t can produce local diffeomorphisms of neighborhoods of points that are *far* from each other in a same CR-orbit. If $X = (X_1, \dots, X_m)$ is an element of \mathbf{X}^m such that for $t = (t_1, \dots, t_m) \in \mathbf{R}^m$, the map $z \rightarrow X_{m,t_m} \cdots X_{1,t_1} z$ is well defined in a neighborhood of z , we will still denote it for convenience by X_t or Φ (cf. Proposition 1.2).

Let $\Delta_{\mathbf{X}}$ be the *distribution spanned by \mathbf{X}* , i.e. the mapping which to $z \in M$ assigns the linear hull of vectors $X(z)$ where X belongs to \mathbf{X} : it is just the distribution associated with the complex tangent bundle of M . We let $P_{\mathbf{X}}$ denote the smallest distribution which contains $\Delta_{\mathbf{X}}$ and is invariant under complex-flow

diffeomorphisms, or for short the smallest \mathbf{X} -invariant distribution which contains $\Delta_{\mathbf{X}}$. Precisely, $P_{\mathbf{X}}(z)$ is the linear hull of vectors of the form $dX_t(v)$ where $v \in \Delta_{\mathbf{X}}(z')$ and $z = X_t z'$. A C^1 distribution P on M has the *maximal integral manifold property* if for every $z \in M$ there exists a submanifold S of M such that $z \in S$ and for every $z' \in S$, $T_{z'} S = P(z')$. Moreover, S is said to be a *maximal integral manifold* of P if S is an integral manifold of P such that every connected integral manifold of P which intersects S is an open submanifold of S .

Then the results of Sussmann, which extend to the C^2 case tell us that $\mathcal{O}[z]$ is a (connected) maximal integral submanifold of $P_{\mathbf{X}}$ (perhaps with a finer topology) and admits a unique differentiable structure making the injection $i : \mathcal{O}[z] \rightarrow M$ an immersion of class C^1 .

We now introduce the following definitions.

DEFINITION 3.3. Let M be a generic submanifold of \mathbf{C}^n and $z \in M$. M is called *minimal at z* if $\mathcal{O}^{loc}[z]$ contains a neighborhood of z in M . It is called *globally minimal at z* if $\mathcal{O}[z]$ contains a neighborhood of z .

In view of the global results of Sussmann definition 3.3 means that the generic manifold M is globally minimal at a point z if and only if there exist a finite number of points z'_l , $l = 1, \dots, d$ in the CR-orbit of z and composed flow diffeomorphisms Φ'_l , $l = 1, \dots, d$ of a neighborhood of z'_l in M on a neighborhood of z in M respectively such that

$$T_z M = \sum_{l=1}^d d\Phi_l(z'_l) \cdot (T_{z'_l}^c M).$$

We are now able to prove the theorem conjectured by Trépreau in [61] which is the natural generalization of a celebrated theorem of Tumanov ([67]). Here is the substance of this paper.

THEOREM 3.4. Let M be a generic submanifold of \mathbf{C}^n of smoothness class $C^{(k, \alpha)}$, $k \geq 2, 0 < \alpha < 1$ which is globally minimal at a point $z \in M$. Then for every z' in the CR-orbit of z there exists a wedge \mathcal{W} of edge M at z' such that

(*) every CR function on M extends holomorphically into \mathcal{W} .

PROOF. We shall make use of the following abuse of language : we will say that a CR-function u is CR-extendible in the direction $v \in TM/T^c M[z]$ if it is in fact CR-extendible in the direction of Jv . Let us consider the set

$$H_z = Vect \quad \{v \in T_z M/T_z^c M; \quad u \text{ is CR-extendible at } (z, v) \}$$

and its preimage under the natural surjection $\pi : TM \rightarrow TM/T^c M$

$$\hat{H}_z = \pi^{-1}(H_z) \subset T_z M.$$

LEMMA 3.5. Let X be a C^1 section of $T^c M$ over a neighborhood of $z \in M$ and let Φ_t be the flow of X and $\Phi = \Phi_t$ for some t . Then, if $v \in T_z M$,

$$v \in \hat{H}_z \iff d\Phi(z).v \in \hat{H}_{\Phi(z)}.$$

PROOF. Since the statement is a symmetric and a transitive one we can assume that z and z' are so close that $z' := \Phi(z)$ is contained in a CR submanifold S of M with $CRdim S = CRdim M$ which is minimal at z (for instance take for S the local CR-orbit of z) and such that z' belongs to the boundary of the manifolds whose existence comes from Theorem 3.1. Hence

$$(*) \quad \hat{H}_{z'} \supset T_{z'} S.$$

So if v belongs to $T_z S$ there is nothing to add. On the other hand, if $\xi = pr_{T_S M} v \neq 0$ we apply the propagation result Theorem 3.2 and obtain that for every $\epsilon > 0$ u is CR-extendible at (z', ξ'') , where ξ'' is ϵ -close in euclidean norm to $\xi' = d\Phi(x).\xi$; so letting ϵ decrease to zero, since every finite-dimensional vector space is closed, we have $\xi' \in pr_{T_S M} (H_{z'})$. Because of (*) the indetermination on the specific representative of ξ' is removed whence

$$d\Phi(z).v \in \hat{H}_{z'}$$

and the lemma is proved.

END OF PROOF OF THEOREM 3.4. The global lemma 3.5 and the condition of global minimality implice immediately that

$$\hat{H}_{z'} = T_{z'} M$$

for every z' in the (global) CR-orbit of z . The conclusion follows by the edge-of-the-wedge theorem and the proof is complete.

Theorem 4.1 admits an obvious generalization which involves the concept of \mathcal{W}_r -wedges. Recall that a \mathcal{W}_r -wedge at z with edge M is locally the general intersection of a wedge of edge M at z and a generic manifold containing M as a submanifold of codimension r .

THEOREM 3.6. *Let M be a generic submanifold of \mathbf{C}^n of smoothness class $C^{(k,\alpha)}$, $k \geq 2, 0 < \alpha < 1$, and let $r = \dim \mathcal{O}[z] - 2CR \dim M$. Then for every z' in the CR-orbit of z , every γ with $0 < \gamma < \alpha$, there exists a \mathcal{W}_r -wedge \mathcal{W} of edge M at z' and of smoothness class $C^{k,\gamma}$ such that*

(*) every CR function on M extends to be CR on \mathcal{W} .

Moreover, the tangent space to \mathcal{W} at z' spans $T_{z'} M + JT_{z'} \mathcal{O}[z]$.

PROOF. The same argument runs in proving that $\hat{H}_{z'}$ contains $T_{z'} M + JT_{z'} \mathcal{O}[z]$ and the conclusion then follows by the edge-of-the wedge theorem of Ayrapetyan ([4]), in the classes $C^{(k,\alpha)}$, $k \geq 2, 0 < \alpha < 1$.

References

- [1] P. AHERN and W. RUDIN, *Totally real embeddings of S^3 into \mathbb{C}^3* , Proc. Amer. Math. Soc. **94** (1985), 460-462.
- [2] H. ALEXANDER, *A note on polynomially convex hulls*, Proc. Amer. Math. Soc. **33** (1972), 389-391.
- [3] S. ALINHAC et P. GÉRARD, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, InterEditions/Editions du CNRS, 1991.
- [4] R. A. AYRAPETIAN, *Extending CR functions from piecewise smooth CR manifolds*. Mat. Sbornik **134** (1987), 108-118. Trad. in english in Math. USSR Sbornik **62** (1989), 1, 111-120.
- [5] R. A. AYRAPETIAN and G. M. HENKIN, *Analytic continuation of CR functions through the “edge of the wedge”*, Dokl. Akad. Nauk. SSSR **259** (1981), 777-781.
- [6] M. S. BAOUENDI AND L. P. ROTHCHILD, *Cauchy-Riemannian functions on manifolds of higher codimension in complex space*. Invent. Math. **101** (1990), 45-56.
- [7] M. S. BAOUENDI, L. P. ROTHCHILD AND J.-M. TRÉPREAU, *On the geometry of analytic discs attached to real manifolds*, Preprint 1992.
- [8] M. S. BAOUENDI and F. Treves, *A property of the functions and distributions annihilated by a locally integrable system of vector fields*, Ann. of Math. **114** (1981), 387-421.
- [9] R. F. BASENER, *Peak points, barrier and pseudoconvex boundary points*, Proc. Amer. Math. Soc. **65** (1977), 89-92.
- [10] E. BEDFORD, *Locally determined envelopes of holomorphy in \mathbb{C}^2* , Math. Ann. **281** (1988), 419-429.
- [11] S. BERHANU and G. A. MENDOZA, *Orbits and global unique continuation for systems of vector fields*, Preprint 1991.
- [12] E. BISHOP, *Differentiable manifolds in complex euclidean space*, Duke Math. J. **32** (1965), 1-22.
- [13] J.-M. BONY, *Principe du maximum, inégalités de Harnack*, Ann. Inst. Fourier, **19** (1969), 277-304.
- [14] AL. BOGGESS AND J. PITTS, *CR extension near a point of higher type*, C. R. Acad. Sci. Paris Sér. I Math. **298** (1984), 9-12.
- [15] B. CHABAT, *Introduction à l'Analyse Complexe*, Éditions Mir, Moscou, 1990.
- [16] G. DE RHAM, *Variétés différentiables, courants, formes harmoniques*, Hermann, Paris, 1960.
- [17] J. DUVAL, *Un exemple de disque polynômialement convexe*, Math. Ann. **281** (1988) 583-588.
- [18] J. DUVAL, *Convexité rationnelle des surfaces lagrangiennes*, Invent. Math. **104** (1991), 581-599.
- [19] J. DUVAL, *Une contrainte géométrique pour certaines sous-variétés rationnellement convexes*, Math. Ann. **289** (1991), 627-629.
- [20] O. FORSTER, *Lectures on Riemann Surfaces*, Graduate Texts in math. **81**.
- [21] F. FORSTNERIC, *On some totally real embeddings of three manifolds*, Manuscripta Math. **55** (1986), 1-7.
- [22] F. FORSTNERIC, *Analytic discs with boudaries in a maximal real submanifold of \mathbb{C}^n* , Ann. Inst. Fourier, Grenoble **37**, 1 (1987), 1-44.
- [23] F. FORSTNERIC, *A totally real three sphere in \mathbb{C}^3 bounding a family of analytic discs*, Proc. Amer. Math. Soc. **108** (4) (1990), 887-892.
- [24] F. FORSTNERIC, *Complex tangents of real surfaces in complex surfaces*, Duke Math. J. **67** (1992), 353-376.
- [25] F. FORSTNERIC, *A smooth holomorphically convex disc in \mathbb{C}^2 that is not polynomially convex*, Proc. Amer. Math. Soc. **116** (1992), 411-415.
- [26] F. FORSTNERIC, *Approximation of smooth mappings by automorphisms of \mathbb{C}^n* , Preprint 1993.

- [27] F. FORSTNERIC, *Intersection of analytic discs and smooth discs*, Preprint 1993.
- [28] F. FORSTNERIC, *Complement of Runge Domains and holomorphic hulls*, Michigan Math. J., *to appear*.
- [29] F. FORSTNERIC and J. GLOBEVNIK, *Discs in pseudoconvex domains*, Comment. Math. Helv. **67** (1992), 129-145.
- [30] J. GLOBEVNIK, *Boundary interpolation and proper holomorphic maps from the disc to the ball*, Math. Z. **198** (1988), 143-150.
- [31] J. GLOBEVNIK, *Perturbation by analytic discs along maximal real submanifolds of \mathbb{C}^n* , Preprint 1993.
- [32] M. GOLUBITSKY and V. GUILLEMIN, *Stable mappings and their singularities*, Graduate Texts in Math. **14**.
- [33] S. GREENFIELD, *Extendibility properties of real submanifolds of \mathbb{C}^n* , Proceedings of the C.I.M.E. Summer Conference on Bounded Homogeneous Domains, Urbino, 1967.
- [34] M. W. HIRSCH, *Differential Topology*, Graduate Texts in math. **33**.
- [35] C. D. HILL and G. TAIANI, *Families of analytic discs in \mathbb{C}^n with boundaries in a prescribed CR submanifold*, Ann. Scuol. Norm. Sup. Pisa **4-5** (1978), 327-380.
- [36] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators I*, Grundlehren der mathematischen Wissenschaften, **256**.
- [37] N. HANGES and F. TREVES, *Propagation of holomorphic extendibility of CR functions*, Math. Ann. **263** (1983), 157-177.
- [38] N. HANGES and F. TREVES, *On the local holomorphic extension of CR functions*, Preprint 1992.
- [39] P. HONVAULT, *Ensembles de module maximum des domaines rigides de \mathbb{C}^2* , Thèse 1994, Université de Lille I, à paraître dans Math. Z.
- [40] N. E. STENNROD, P. R. HALMOS, M. M. SCHIFFER and J. DIEUDONNÉ, *How to write mathematics*, American mathematical society, 1975.
- [41] A. IORDAN, *Ensemble de module maximal dans des domaines pseudoconvexes*, C. R. Acad. Sc. Paris Sér. **300** (1985), 655-656.
- [42] B. JÖRICKE, *Removable singularities of CR-functions*, Ark. Mat. **26** (1988), 117-143.
- [43] B. JÖRICKE, *Envelopes of holomorphy and CR-invariant subsets of CR-manifolds*, C. R. Acad. Sci. Paris Sér. **1 315** (1992), 407-411.
- [44] B. JÖRICKE, *Deformation of CR-manifolds, minimal points and CR-manifolds with the microlocal analytic extension property* Preprint 1994.
- [45] S. G. KRANTZ, *Function theory of several complex variables*, Wadsworth and Brooks/Cole 1992.
- [46] S. LANG., *Analysis II*, Addison-Wesley, 1969.
- [47] P. LELONG, *Fonctions plurisous-harmoniques et formes différentielles positives*, Gordon and Breach, Dunod, 1968.
- [48] H. LEWY, *On the local caracter of the solution of an atypical differential equation in three variables and a related problem for regular functions of two complex variables*, Ann. of Math. **64** (1956), 514-522.
- [49] H. LEWY, *An example of a smooth linear partial differential equation without solution*, Ann. of Math. (2) **66** (1957), 155-158.
- [50] S. MAINGOT, *Sur l'extension des fonctions CR*, C. R. Acad. Sci. Paris Sér. **1 308** (1990), 291-295.
- [51] M. E. MARSON, *Wedge extendability for hypoanalytic structures*, unpublished.
- [52] J. MERKER, *Global minimality of generic maifolds and holomorphic extendibility of CR functions*, Int. Math. Res. Not. **8** (1994), 329-342.

- [53] A. NAGEL and J.-P. ROSAY, *Maximum modulus sets and reflexion sets*, Ann. Inst. Fourier **41** (1991), 2, 431-466.
- [54] S.I. PINCHUK, *A boundary uniqueness theorem for holomorphic functions of several complex variables*, Mat. Zametki **15** (1974), 205-212; English transl. in Math. Notes **15** (1974).
- [55] J.-P. ROSAY, *Sur un problème d'unicité pour les fonctions CR*, C. R. Acad. Sci. Paris Sér. I **302** (1986), 9-11.
- [56] J.-P. ROSAY, *À propos de "wedge" et d"edge", et de prolongements holomorphes*, Trans. Amer Math. Soc. **297** (1986), 63-72.
- [57] J.-P. ROSAY, *A remark on a theorem by F. Forstneric*, Ark. Math. **28** (1990) 311-314.
- [58] M. SPIVAK *Differential geometry*, volume one, Brandeis university, 1970.
- [59] H. J. SUSSMANN, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. **180** (1973), 171-188.
- [60] J.-M. TRÉPREAU, *Sur le prolongement holomorphe des fonctions CR définies sur une hypersurface réelle de classe \mathbf{C}^2* , Invent. Math. **83** (1986), 583-592.
- [61] J.-M. TRÉPREAU, *Sur la propagation des singularités dans les variétés CR*, Bull. Soc. Math. Fr. **118** (1990), 403-450.
- [62] J.-M. TRÉPREAU, *On the extension of holomorphic functions across a real hypersurface*, Math. Z. **211** (1992), 93-103.
- [63] J.-M. TRÉPREAU, *On the Global Bishop Equation*, Unpublished.
- [64] J.-M. TRÉPREAU, *Cours de DEA, 2^d Semestre 1992*, Manuscrit.
- [65] F. TREVES, *Basic linear differential equations*, Academic Press, New York, 1975.
- [66] F. TREVES, *Approximation and representation of functions and distributions annihilated by a system of complex vector fields*. Palaiseau, Ecole Polytechnique, Centre de Mathématiques, 1981.
- [67] A. E. TUMANOV, *Extending CR functions on a manifold of finite type over a wedge*, Mat. Sbornik **136** (1988), 129-140. Trad in english in Math. of the USSR Sbornik **64** (1989), 1, 129-140.
- [68] A. E. TUMANOV, *Extending CR-functions into a wedge*. Mat. Sbornik **181** (1990), 951-964. Trad. in English in Math. USSR Sbornik **70** (1991), 2, 385-398.
- [69] A. E. TUMANOV, *Connections and propagation of Analyticity for CR Functions*, Preprint 1992. To appear in Duke Math. J.
- [70] S. M. WEBSTER, *The Euler and Pontrjagin numbers of an n -manifold in \mathbf{C}^n* , Comment. Math. Helv. **60** (1985) 193-216.
- [71] S. M. WEBSTER, *On the reflection principle in several complex variables*, Proc. Amer. Math. Soc. **71** (1978), 26-28.
- [72] R. O. WELLS, *Differential Analysis on Complex Manifolds*, Graduate Texts in Math. **65**, Springer, New-York-Heidelberg-Berlin, 1993.
- [73] R. O. WELLS, *Compact real submanifolds of a complex manifold with nondegenerate holomorphic tangent bundles*, Math. Ann. **179** (1969), 123-129.
- [74] J. WERMER, *Polynomially convex discs*, Math. Ann. **158** (1965), 6-10.
- [75] J. WERMER, *Banach Algebras and Several Complex Variables*, Graduate Texts in Math. **35**.
- [76] E. C. ZACHMANOGLU, *Propagation of the zeroes and Uniqueness in the Cauchy Problem for first order partial differential equations*, Arch. Rat. Mech. Anal. **38** (1970), 178-181.
- [77] E.C. ZACHMANOGLU, *Foliations and the propagation of zeroes of solutions of partial differential equations*, Bull. Amer. Math. Soc. **78** (1972), 835-839.